# 特異積分方程式の数値解法 : 除く, 固有値問題, Volterra型, 微積分方程式, 純理論 (特異積分方程式の数値解法)

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# 特異積分方程式の数値解法

(除く. 固有値問題, Volterra型, 微積分方程式, 純理論)

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## §1 方程式の分類

$$\begin{cases} \text{第1種} & \int K(s,t)f(t)\,dt = g(s). \\ \text{第2種} & f+\int K(s,t)f(t)\,dt = g(s). \\ \text{同次} & \lambda\int Kf = f. \end{cases}$$

$$\begin{cases} \textit{regular} & \text{核 } K: \text{自乗積分可能}. \\ \textit{singular} & K: \text{自乗積分発散}, \ etc. \end{cases}$$

例 $K=H(s,t)(s-t)^{-n}$, $K=H(s,t)\cot(s-t)$, ? $K=\ln|s-t|$

## §2 教科書

実用的

1) 中田義元, 加藤敏夫, ···, "微分方程式の近似解法 I"
岩波講座, 現代応用数学.

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Oxford $ 7.00. 1952.

(方程式 $J(\tau)=\frac{1}{2}\int_0^\infty J(t)E_1(|t-\tau|)dt$ について)

## §3 積分方程式の解法

1) 厳密解，解析的方法．(参考：教科書，文献 6．48)-76))

例．Abel's eq. $\int_0^s \frac{f(t)}{\sqrt{s-t}}dt = g(s)$.

解： $f(t)=\frac{1}{\pi}\frac{d}{dt}\int_0^t \frac{g(s)}{\sqrt{t-s}}ds$.

Cauchy 核をもつ eq.

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{f(t)}{s-t}dt = g(s).$$

解: $$f(t)=\frac{1}{\pi}\frac{1}{\sqrt{1-t^2}}\left[C-\int_{-1}^{1}\frac{g(s)}{t-s}\sqrt{1-s^2}\,ds\right]$$

Wiener–Hopf の方程式

2) 近似解法 (正規核及び特異核)

(i) 特異核を正規な核に変換して，通常の近似法を用いる。(参考：§2 の教科書, 文献 5. の 44)–47).)

例 Hamel (教科書 5) P 140).

Bückner (教科書 3) pp 443–444).

(ii) 核を縮退核で近似。(参考：§2 の教科書 1), 3), 4), 文献 29)–31).)

例 Bateman の方法，直交関数展開，Galerkin 法。

(iii) 逐次近似法 iterative methods.

(参考：教科書 1), 3), 4) etc. 文献 1. の 1)–23).)

例 第 2 種方程式に対して

Neuman–Liouville, Schmidt, Wiarda,

Bückner, Schönberg, Bellman, Wagner, …,

第 1 種方程式に対して

Berg, … .

(iv) 変分法

(v) Analog methods, 数値積分公式または補間公式を使う。(参考：教科書, 中田 1) pp 28–32, P 40, 日高 2), 文献 4 の 32)–43).)

$$\int K(s,t)f(t)\,dt \fallingdotseq \sum K_{ij} f(t_j).$$

特異方程式に対して,

(イ) Kryloff の方法（中田 1) P40).

$$K=(s,t)=\frac{d}{dt}H(s,t) \text{ とすると}$$

$$\int Kf\,dt=\int f\,dH \fallingdotseq \sum f(t_n)\Delta H$$

(ロ) Nyström の方法（日高 2))。

補間公式を用いる。煩雑。

(ハ) Sofronov（文献 35）または Math. Rev. 19 (1958) p. 66.)

(ニ) 簡便な方法（中田 1) p31).

$$\lim_{t\to s} K(s,t)(t-s)=0 \text{ のとき}$$

$$\int K(s,t)f(t)\,dt=\int K(s,t)\{f(t)-f(s)\}\,dt + f(s)\int K(s,t)\,dt.$$

右辺の初項は通常の数値積分公式を使う。第2項はKが解析的に積分できないときは,

$$K=\mathcal{K}+(K-\mathcal{K})$$

とすると（$\mathcal{K}$：Kの principal part, 初等関数）

$$\int K\,dt=\int \mathcal{K}\,dt+\int (K-\mathcal{K})\,dt$$

(ホ) Young, (Aso),

$$K=\mathcal{K}_0 P+(K-\mathcal{K}_0 P),$$

P：多項式（または正則関数）

数値積分公式 $\int K_0 F(t)dt = \sum_j w_j(s) F(t_j)$.

$$\int Kf\,dt = \int K_0 Pf\,dt + \int (K - K_0 P) f\,dt .$$

$K_0 P$ の選択に注意.

$\begin{cases} \text{特異点近傍で K のよい近似.} \\ \text{特異点から離れた所でも K との差は大きくならない。} \end{cases}$

失敗例 ‥‥‥ § 4。

(vi) Monte Carlo 法

§ 4. 計算失敗例.

流れに垂直な平板を過ぎる粘性流（Oseen 近似）

$l = R/4$, $(k = U/2\nu = 1,\ R = 2lU/\nu = 4kl)$.

Eq. $\int_{-l}^{l} [K_0(y-\eta)F(y) + iH(y-\eta)\bar{F}(y)]dy = 4\pi U$.

ただし $H(Y) = (1/Y) - K_1(Y)$,

$K_0, K_1$ : 変形ベッセル関数.

$F(y)$ は $y = \pm l$ で $1/\sqrt{l^2 - y^2}$ なる形の特異性を持つ。

よって

変換: $y = l \sin t$, $\eta = l \sin \tau$,

$$f(t) = l \cos t \cdot F \equiv f_1 + i f_2 .$$

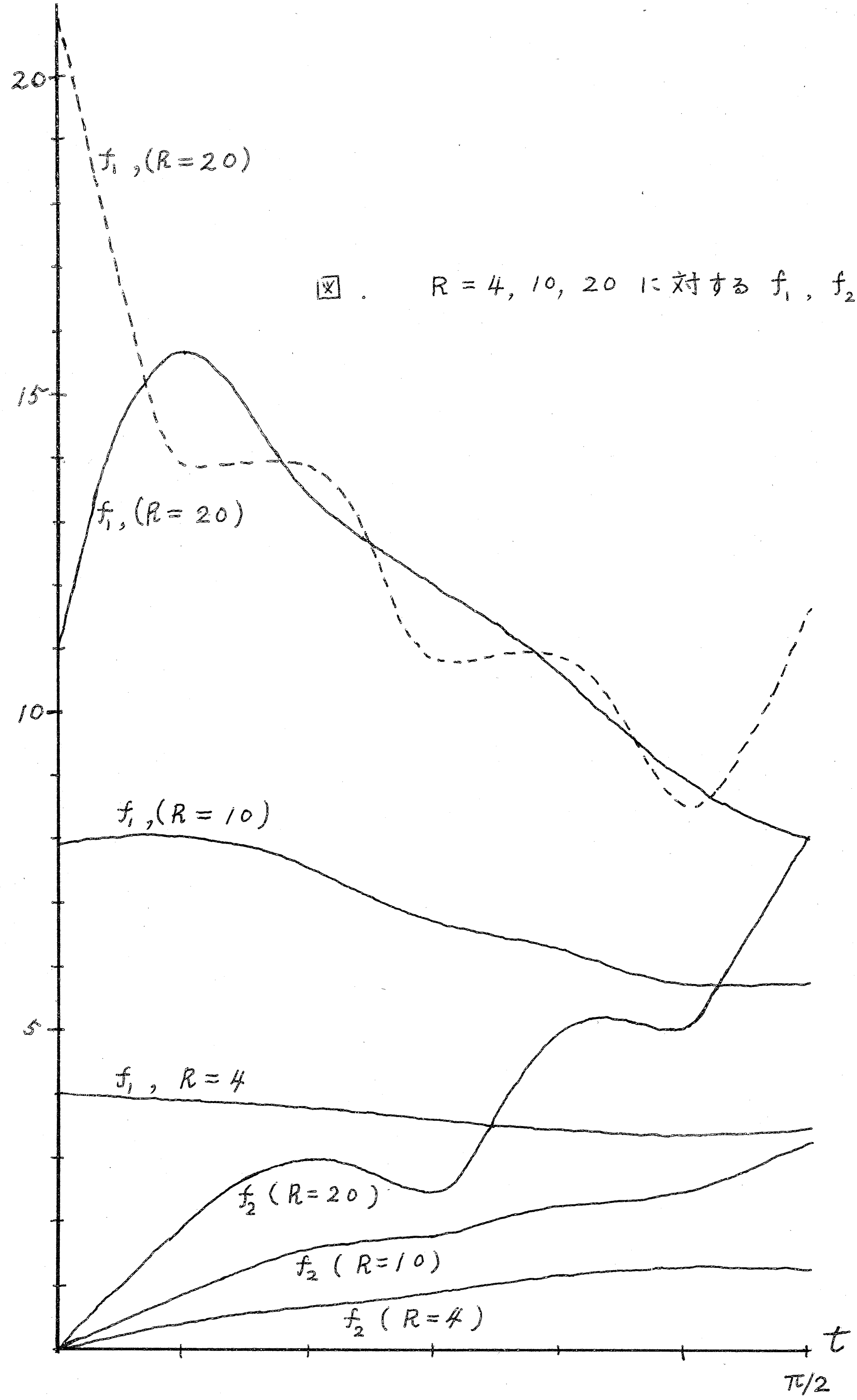
数値解 —— § 3 (v)(ホ) の方法による。

数値積分公式　$\int_{-1}^{1} f(x) \ln |x-\xi_m| dx = \sum_n w_n(\xi_m) f(x_n)$,

係数 $w_n(\xi_m)$はあらかじめ求めてある。
$f_1(t)$, $f_2(t)$はそれぞれ偶，奇関数であるので積分域を[0, π/2]とする。更に変換($t, \eta \to x, \xi$)して[-1, 1]とする。

$$K_0(y-\eta) = \ln 2 - \gamma - \ln\left(\frac{\pi}{4} l \cos \tau\right)$$
$$- \ln|x-\xi| \left[ 1 ⌟ + \frac{1}{4} l^2 \left\{ \cos^2 \tau (t-\tau)^2 - \cos\tau \sin\tau (t-\tau)^3 \right\} \right]$$
$$+ \cdots$$
$$H(y-\eta) = \cdots ⌟ \cdots$$

そこで$K_0P$として上式の⌟まで，または終までを採用する。前者による計算結果は図の実線，後者によるものは点線をもって示されてある。区間[-1, 1]は6等分した。後者の方が，労多くしてかえって悪い結果に導いた。ただし，低Reynolds数では，両者は一致した。Yが大なるとき$K_0(Y) \to 0$であるのに，$\ln|Y| \cdot Y^3$等はかなり大となり，$\int (K_0 - K_0P) f dt$の評価において誤差を大きくしたものと思われる。




図． R＝4, 10, 20 に対する $f_1$, $f_2$ の値

## 5.文献

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(ii) 次に関するものはできるだけ除いたが, 厳密ではない。固有値問題,Volterra 型, 微積分方程式, 純理論, ロシヤ語の論文.

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$$f_{n+1} = f_n - E(g, f_n, K) - j\,E(g, f_n, K), \quad E(g, f, K) \equiv g - f - \int K f.$$

$j = 0$ ととると, Neuman series.

$j = \text{const.} \neq 0$, Wiarda, Bückner.

$j = -\bar{K}(s)/[1+(b-a)\bar{\bar{K}}]$, C. Wagner の方法,

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eq. $\int_{-a}^{a}\int_{-b}^{b} f(\xi,\eta)[(x-\xi)^2+(y-\eta)^2]^{-1/2} d\xi d\eta = g(x,y)$,

maximum of

$$I(F) = \lambda\iint F\,dx\,dy - \iiiint F(\xi,\eta)F(x,y)[(x-\xi)^2+(y-\eta)^2]^{-1/2} dx\,dy\,d\xi\,d\eta,$$

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$$\begin{cases} K(s,t) \longrightarrow H_n(s,t) \\ H_n(s,t) = \Sigma\Sigma\alpha_{ij} K(s,t_j)K(s_i,t)/W_n, \quad W_n = \det(K(s_i,t_j)) \end{cases}$$

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$f(s)-\lambda\int_a^b Kf\,dt = g(s)$, 求積法により $\int_a^b F(t)dt \fallingdotseq \sum_{j=1}^{n} A_j F(t_j)$.

$\therefore \int K(s_i,t)f(t)dt \fallingdotseq \sum L_{ij} f_j$, 近似解のベクトルを $\tilde{f} = (f_j)$ とする。

ベクトル $\rho \equiv (I-\lambda L)\tilde{f} - g$, ($I$: 単位行列)

Put $\varepsilon_g(s) \equiv \int Kg - \sum A_j K(s,t_j)g(t_j)$, $\varepsilon(s,t) \equiv \int KK - \sum A_j K(s,t_j)K(t_j,t)$

$\varepsilon_1 \equiv \max_{a\le x\le b} \sum_j^n |A_j \varepsilon(s,t_j)|$, $\varepsilon \equiv \max_{1\le i\le n} \int_a^b |\varepsilon(t_i,t)|dt$,

$K_1 \equiv \max_{a\le s\le b} \sum_j |A_j K(s,t_j)|$, $R(s,t,\lambda) \equiv K(s,t)$ の

$\max \int |R(s,t,\lambda)|dt \leqq \Gamma$, $B \equiv 1+|\lambda|\Gamma$, $g \equiv \lambda^2 \varepsilon_1 B \leq 1$,

$\|f-\tilde{f}\| \leqq (|\lambda| \|\varepsilon_g\| + |\lambda|^2 B \|f\| \varepsilon + \|\rho\|)(1+|\lambda| K_1 B)/(1-g)$.

従来の誤差評価よりも精密である。ただし煩雑.


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domain 1) rectangle 2) half-plane 3) strip 4) half-s
5) angle 6) circle 7) half-circle

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eq. $\frac{1}{\pi}\left[\int_{-1}^{-k}+\int_{1}^{k}\right]\frac{\varphi(y)}{y-x}dy=f(x)$, $k<x<1$.

sol. $\varphi=g-\left\{\frac{1}{\pi}\int_{-k}^{k}\left[(1-y^2)(k^2-y^2)\right]^{1/2}\frac{g(y)}{y-x}dy+C_1x+C_2\right\}$,

where $C_1$, $C_2$ : arbit. consts.

$g(x)=-\frac{1}{\pi}\left(\int_{-1}^{-k}+\int_{k}^{1}\right)\left(\frac{1-y^2}{1-x^2}\right)^{1/2}\frac{f(y)}{y-x}dx$.

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eq. $u(x,y)=\iint_S p(\xi,\eta)r^{-1}d\xi d\eta$, $S=\{x^2+y^2\leqq a^2\}$.

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eq. 53) $\int_{-1}^{1}f(t)(1-t^2)^{-1/2}\ln|t-s|dt=\lambda g(s)$, および

$\frac{d^2}{ds^2}\int_{-1}^{1}f(t)(1-t^2)^{1/2}\ln|t-s|dt=\lambda g(s)$.

54) $f(s)=-\frac{1}{2\pi}\int_0^{\pi}g(t)\ln\frac{1-\cos(s+t)}{1-\cos(s-t)}dt$

$g(t)=\frac{1}{2\pi}\int_0^{\pi}f''(s)\ln\frac{1-\cos(s+t)}{1-\cos(s-t)}ds$.

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eq. $g(x)=\int_{-1}^{1} f(y) Z_0(k|x-y|)dy,$

$$g(x)=\int_{-1}^{1} f(y) \frac{|x-y|}{x-y} Z_1(k|x-y|)dy,$$

ただし $Z_n(\alpha)=AJ_n(\alpha)+BN_n(\alpha)$.

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eq. $f(s)=\frac{1}{2}s\int_0^\infty E_1(|s-t|)f(t)dt+g(s),$

$E_n$: the n-th expon. integ.

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eq. $\begin{cases} f(s)+\int_0^\infty K(s-t)f(t)dt = g(s) & s>0 \\ f(s)+\int_0^\infty K(s+t)f(t)dt = g(s) & s<0 \end{cases}$

ただし $K(x)=O(\ln|x|) \quad (x\to 0)$.

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eq. $f(s)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_1(s-t)f(t)\,dt = g(s) \quad (0<s<\infty)$

$$f(s)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_2(s-t)f(t)dt = g(s) \quad (-\infty<s<0).$$

solut. $f(s)=\frac{1}{\sqrt{2\pi}}\int_{ib_1-\infty}^{ib_1+\infty}\Phi^+(\zeta)e^{-is\xi}d\xi-\frac{1}{\sqrt{2\pi}}\int_{ia_2-\infty}^{ia_2+\infty}\Phi^-(\zeta)e^{-is\xi}d\xi.$

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eq. $$\begin{cases} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_1(s-t)f(t)\,dt = g(s) & 0 < x < \infty \\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k_2(s-t)f(t)\,dt = g(s) & -\infty < x < 0 . \end{cases}$$

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